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Initial simplicial complexes of prime ideals

John Dalbec

Youngstown State University, One University Plaza, Youngstown, OH 44555, USA

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Abstract

We provide a two-parameter family of examples of irreducible projective algebraic varieties whose initial complexes (in the sense of [M. Kalkbrener, B. Sturmfels, Adv. Math. 116 (1995) 365–376]) have the maximum number of simplices given the dimensions of the variety and of its ambient projective space. This shows that irreducibility fails to be preserved in the worst possible fashion by the operation of passing to the Stanley–Reisner variety of the initial complex.

Keywords: Initial ideal; Simplicial complex; Stanley–Reisner ideal; Bertini theorem; Cohen–Macaulay varieties; Set-theoretic complete intersection; Symmetric polynomial

1. Introduction

Let I be an ideal in a polynomial ring $R = k[x_0, \dots, x_n]$ and let $<$ be a term order on R. The radical of the initial ideal of I (that is, $\sqrt{\mathrm{in}_<(I)}$) is a square-free monomial ideal; therefore it is the Stanley–Reisner ideal of a simplicial complex. Following [9], we call this simplicial complex the *initial complex* of I and denote it by $\Delta_<(I)$.

In [9], Kalkbrener and Sturmfels show that the initial complex of a prime ideal is pure-dimensional and strongly connected, proving a conjecture of Kredel and Weispfenning [10]. Kalkbrener and Sturmfels illustrate their results with three families of prime ideals: projective toric ideals, determinantal ideals, and ideals of relations of Hodge alge-

E-mail address: jpdalbec@ysu.edu.

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bras. It is natural to ask which complexes can arise as the initial complex of a prime ideal; in the examples above, the initial complex is a topological ball.

Our first result shows that these examples are not the norm; in particular, initial complexes of prime ideals can have arbitrarily large integral homology groups.

Theorem 1.1. *Given an algebraically closed field K, there exists a doubly infinite family of prime ideals $I(X_n^d) \subset K[x_0, \ldots, x_n]$ for $n \geqslant d \geqslant 1$ whose initial complex $\Delta_<(I(X_n^d))$ is the d-skeleton Δ_n^d of the n-simplex.* (*The term order $<$ depends on n.*)

The varieties that we construct in this theorem are manageable only because they are complete intersections. They would not be manageable otherwise because, in general, they are not rational and have large genera.

Our second result is a method of constructing additional examples:

Theorem 1.2. *Let X be a variety, w a weight vector, and $<$ a term order such that $\mathrm{in}_w(I(X)) = \mathrm{in}_<(I(X))$. Given two vertices v_1 and v_2 of $\Delta_<(I(X))$ such that $\{v_1, v_2\}$ is a nonface and such that the variables corresponding to v_1 and v_2 have the same weight with respect to w, there exists a variety Y and a weight vector w' such that for any term order $<'$ refining w', $\Delta_{<'}(I(Y))$ is obtained from $\Delta_<(I(X))$ by identifying the vertices v_1 and v_2.*

This construction is in fact useful. For example, as a corollary we have:

Corollary 1.2.1. *There exist unirational varieties with topologically noncontractible initial complexes. In particular, there exists a unirational variety, one of whose initial complexes is the icosahedral triangulation of* $\mathbf{P}^2$ (*Example* 3.1.2).

We note that Taylor [13] has independently constructed irreducible varieties and term orders that have an arbitrary Cohen–Macaulay initial complex of codimension 2 in the ambient projective space. Also, Herzog, Trung, and Valla present a family of examples of Cohen–Macaulay (in fact Gorenstein) codimension 3 Pfaffian ideals and term orders in [7]. The initial complexes of these ideals are dual to a union of cones over a tower of complete bipartite graphs $K_{1,1} \subset K_{2,2} \subset \cdots \subset K_{r,r}$. The examples presented here are Cohen–Macaulay as well (in fact, most of them are complete intersections), but of arbitrarily high codimension.

2. The complete intersection construction

In [1], it was shown that there is an irreducible curve $C \subset \mathbf{P}^3$ and a term order $<$ on its coordinate ring $K[w, x, y, z]$ such that $\Delta_<(I(C))$ is the edge complex (or 1-skeleton) of a tetrahedron. In this section, we prove Theorem 1.1, which generalizes this example to construct, for each (d, n) with $1 \leqslant d \leqslant n$, an irreducible d-dimensional subvariety $X_n^d \subset \mathbf{P}^n$ and a term order $<$ on $K[x_0, \ldots, x_n]$ such that $\Delta_<(I(X_n^d))$ is the d-skeleton of an n-simplex. (Note: If $d = 0$, then any irreducible variety consists of a single point and its initial complex is necessarily a one-point complex. Therefore no irreducible variety X_n^0 can satisfy the conclusion of Theorem 1.1 unless $n = 0$.)

We will construct defining polynomials for the variety X_n^d with explicitly defined monomials, although the (nonzero) coefficients of these monomials will simply be shown to exist. We fix d and n.

Our construction of the defining polynomials has four steps. Step (3) in particular depends on the notion of the *initial form* of a polynomial p with respect to a weight vector w, denoted $\mathrm{in}_w(p)$. We define this to be the sum of the terms having the highest weight with respect to the weight vector. For example, the initial form of the polynomial $x^3 + 2xyz + 3y^3$ with respect to the weight vector $(0, 1, 2)$ is $2xyz + 3y^3$. Note that the coefficients cannot be dropped as one can drop the coefficient in an initial term. We sketch the four steps of the construction below, and then fill in the technical details.

(1) (Proposition 2.1) Let A_n^d be the arrangement of all coordinate d-planes in $\mathbf{P}^n$. Write A_n^d as a set-theoretic complete intersection of homogeneous polynomials f_i, in such a way that the exponents can be freely altered without changing the underlying reduced scheme. The fact that $x_i \mid f_i$ in our chosen polynomials is important to the success of step (2).

(2) (Proposition 2.2) Choose a weight vector w and deform the exponents of the f_i to get homogeneous polynomials $x_i^{r_i+1} g_i$, all of whose terms have the same weight with respect to the chosen weight vector w. These polynomials still define A_n^d set-theoretically since the monomial supports have not changed.

(3) (Proposition 2.3) Add new monomials to the polynomials $x_i^{r_i+1} g_i$ to get the polynomials h_i; the terms of each h_i define a monomial map that is an isomorphism on the complement of a subvariety of $\mathbf{P}^i$ of dimension $d - 1$. The initial forms of the h_i with respect to w still define A_n^d set-theoretically, and we show that the ideal of initial forms of $\langle\{h_i\}\rangle$ also defines A_n^d set-theoretically. Furthermore, it is easy to see that the initial ideal of $\langle\{h_i\}\rangle$ with respect to $<$ (which we choose so that it refines w) must define A_n^d set-theoretically since $I(A_n^d)$ is already a monomial ideal.

(3) (Proposition 2.4) Here we proceed by induction on n; d is still fixed. Assign “generic” coefficients (in fact, the coefficients will be in our infinite ground field k) to the monomials in h_i to get p_i. The polynomials p_i define X_n^d. The initial forms (and hence the initial ideal) of the p_i still define A_n^d set-theoretically since only the coefficients have changed. Finally, the choice of coefficients allows us to show that X_n^d is irreducible.

Proposition 2.1. *The variety A_n^d of the Stanley–Reisner ideal of Δ_n^d is a set-theoretic complete intersection.*

Proof. Since the variety A_n^d consists of all coordinate d-planes in $\mathbf{P}^n$, a point of $\mathbf{P}^n$ belongs to A_n^d if and only if at most $d+1$ of its coordinates are nonzero. Let e_k be the kth elementary symmetric function of its arguments. Let

$$f_i = e_{d+1}(x_0, x_1, \dots, x_{i-1})x_i \tag{1}$$

for $i = d+1, \ldots, n$ and let $Y_n^d = V(f_{d+1}, \ldots, f_n)$. Let $\mathbf{a} = (a_0, \ldots, a_n)$ be a point in $\mathbf{P}^n$. Suppose that at most $d+1$ of the a_j are nonzero; then every monomial in f_i is zero at $\mathbf{a}$ because its support consists of $d+2$ of the variables, one of which must be zero at $\mathbf{a}$. Thus each f_i is zero at $\mathbf{a}$ and so $\mathbf{a} \in Y_n^d$. Conversely, suppose that $d+2$ or more of the a_j are nonzero, and fix j to be the index of the $(d+2)$th nonzero a_j. The only nonzero monomial in f_j is the product of the first $d+2$ nonzero a_j. Therefore, f_j is nonzero at $\mathbf{a}$ and $\mathbf{a} \notin Y_n^d$. We conclude that $A_n^d = Y_n^d$ set-theoretically. Now Y_n^d is defined by $n-d$ polynomials and has codimension $n-d$, so it is a complete intersection. $\square$

Remark 2.1.1. The fact that the polynomials f_i in the proof of Proposition 2.1 define A_n^d set-theoretically depends only on their monomial supports. Therefore these polynomials will still define A_n^d set-theoretically if we modify each polynomial by assigning to each of its terms an arbitrary nonzero coefficient and arbitrary nonzero exponents on the variables in its support.

Proposition 2.2. *Given an increasing weight vector $w = (w_0, \ldots, w_n) \in \mathbf{Z}^{n+1}$, we construct for each i a polynomial g_i in the variables $y_j = x_j/x_i$ for $0 \leqslant j < i$. For $r \gg 0$ we may consider $x_i^r g_i$ as a polynomial in the variables x_j for $0 \leqslant j \leqslant i$, and this polynomial has the following properties*:

- *The monomials of f_i and $x_i^r g_i$ are in one-to-one correspondence such that corresponding monomials have the same support in the variables x_j.*
- *The polynomial $x_i^r g_i$ is homogeneous with respect to the weight vector w.*

Remark 2.2.1. In practice we always use the weight vector given by $w_i = i$.

Remark 2.2.2. A discussion of how to choose a specific $r \gg 0$ follows the proof.

Proof. Fix i and let $y_j = x_j/x_i$ for $0 \leqslant j < i$. Then y_j has degree 0 in the induced grading and weight $w_j - w_i < 0$. Now for each monomial m in $e_{d+1}(y_0, y_1, \ldots, y_{i-1})$ we choose a corresponding monomial n_m in g_i with the same support as follows: Consider the set W_m consisting of the weights of all monomials in the variables y_j with the same support as m. Note that some weights may arise from more than one monomial. If $m = y_{j_0} \cdots y_{j_d}$, then

$$W_m = \left\{ w\left(y_{j_0}^{l_0} \cdots y_{j_d}^{l_d}\right) \mid l_0, \ldots, l_d \in \mathbf{Z}^+ \right\}$$

(note that each l_j is positive). Equivalently,

$$W_m = \left\{ l_0(w_i - w_{j_0}) + \cdots + l_d(w_i - w_{j_d}) \mid l_0, \ldots, l_d \in \mathbf{Z}^+ \right\}.$$

It follows that all sufficiently large negative multiples of the GCD of the weights are in W_m, that is, $-a \cdot \mathrm{GCD}(w_i - w_{j_0}, \ldots, w_i - w_{j_d}) \in W_m$ for $a \gg 0$. Now the intersection of all the sets W_m will contain all sufficiently large negative multiples of the LCM of the various GCD's, so it is nonempty.

To be constructive, we exhibit a specific member of this intersection. Let $\ell = \mathrm{LCM}(w_i - w_0, \ldots, w_i - w_{i-1})$. Then $-(d+1)\ell$ belongs to the intersection because we can assign the exponent $\ell/(w_i - w_j)$ to each variable y_j in the support of m. However, we can often find much smaller negative weights in the intersection. In the example that follows this proof, we will choose the negative weights to be as small as possible.

Choose an element w' of the intersection and for each monomial $m = y_{j_0} \cdots y_{j_d}$ in $e_{d+1}(y_0, \ldots, y_{i-1})$ choose a monomial $n_m = y_{j_0}^{l_0} \cdots y_{j_d}^{l_d}$ of weight w' having the same support as m; let g_i be the sum of the monomials n_m. The desired properties of $x_i^r g_i$ follow by construction. $\square$

By Proposition 2.1 and Remark 2.1.1, it follows from the first property above that the polynomials $x_i^{r_i} g_i$ define A_n^d set-theoretically for r_i sufficiently large. We now choose a specific r_i as follows: Let s be the minimum weight $w(m)$ over all monomials $m \in g_i$, and let t be the maximum degree $\deg_{y_0,\ldots,y_{i-1}}(m)$ over all monomials $m \in g_i$ (where each variable y_j has degree 1). Choose r_i to be the maximum of t and $\lfloor s/(w_{i-d} - w_i) \rfloor + 1$. Of course this is not possible when $d = 0$ because then the second formula would involve division by zero. This is one reason why the statement of the theorem is restricted to $d \geqslant 1$. Taking $r_i > s/(w_{i-d} - w_i)$ will allow us to construct particular polynomials h_i with initial forms $x_i^{r_i+1} g_i$; taking $r_i \geqslant t$ ensures that $x_i^{r_i+1} g_i$ is in fact a polynomial in the variables x_j and has the correct monomial supports. (The variable x_i must appear at least with exponent 1 in each monomial, which is why we use $x_i^{r_i+1} g_i$ and not just $x_i^{r_i} g_i$.)

For each i (recall that $2 \leqslant d + 1 \leqslant i \leqslant n$), let

$$h_i = x_i^{r_i+1} g_i + (x_0 + x_1 + \cdots + x_i)\bigl(x_0^{r_i} + x_1^{r_i} + \cdots + x_{i-d}^{r_i}\bigr). \tag{2}$$

Then the initial form of h_i with respect to w is $x_i^{r_i+1} g_i$ by the first condition on the choice of r_i. These initial forms define A_n^d set-theoretically, and we will show that we can randomize the coefficients of the h_i to obtain polynomials p_i defining an irreducible variety $X_n^d = V(p_{d+1}, \ldots, p_n)$. First, however, we show that the initial complex has not changed.

Proposition 2.3. *Given h_i as above and choosing any term order $<$ refining the weight vector w, let $I = (h_{d+1}, \ldots, h_n)$. Then $\Delta_<(I) = \Delta_n^d$.*

Proof. The initial forms of the h_i define A_n^d, which has dimension d, so the dimension of $V(I)$ is at most d. But I has $n - d$ generators, so it is a complete intersection. Let J be the ideal generated by the polynomials $x_i^{r_i+1} g_i$ for $i = d+1, \ldots, n$. Since h_i and its initial form $x_i^{r_i+1} g_i$ have the same degree, the ideals I and J have the same Hilbert series, because the minimal free resolution of each is the Koszul resolution. Clearly J is contained in $\mathrm{in}_w(I)$, and since their Hilbert series match, equality holds.

By Proposition 2.1 and Remark 2.1.1, J defines the complex Δ_n^d. Since the term order $<$ refines the weight vector w, $\mathrm{in}_<(I) = \mathrm{in}_<(J)$. We leave it as an exercise for the reader to show that

$$\sqrt{\mathrm{in}_<(J)} = \sqrt{\mathrm{in}_<\bigl(\sqrt{J}\,\bigr)}. \tag{3}$$

Since $\sqrt{J} = I(A_n^d)$ is a monomial ideal, we find that $\sqrt{\mathrm{in}_<(I)} = I(A_n^d)$, and the conclusion follows. $\square$

Proposition 2.4. *The variety $X^d = V(p_{d+1}, \dots, p_n) \subset \mathbf{P}^n$ is irreducible, where p_i has the same monomials as h_i but "generic" coefficients.*

We will need the following results in the proof:

Theorem 2.5 (Lasker [2,11,12]). *A complete intersection in projective space has unmixed dimension. In particular, it has no embedded components.*

Theorem 2.6 (Bertini [4,6,8]). *Let k be an infinite field and let X be a geometrically irreducible subvariety of $\mathbf{P}_k^m$. If $\dim(X) \geqslant 2$, then there is a nonempty Zariski-open set $U \subset \mathbf{P}_k^{m*}$ such that for all hyperplanes $L \in U$, $X \cap L$ is geometrically irreducible.*

Remark 2.6.1. The statement of Theorem 2.6 is the special case $d = 1$ of Corollary 6.11, part (3) in [8]. We may shrink U in the conclusion of Theorem 2.6 so that none of the coefficients defining the hyperplane L are zero.

Proof. We now prove Proposition 2.4 (thus completing the proof of Theorem 1.1) by induction on n, keeping d fixed. The base case for the induction is $X_d^d = \mathbf{P}^d$, which is clearly geometrically irreducible and is its own initial complex.

Suppose that $X_m^d = V(p_{d+1}, \dots, p_m) \subset \mathbf{P}^m$ is geometrically irreducible and that p_i has the same monomials as h_i but arbitrary nonzero coefficients. Embed X_m^d into $\mathbf{P}^{m+1}$ using the first $m+1$ coordinates and let Y_m^d be the projective cone over X_m^d with vertex $(0 : \dots : 0 : 1)$. Given the polynomial h_{m+1} with

$$M_{m+1} + 1 = \binom{m+1}{d+1} + (m+2)(m+2-d)$$

terms, we construct the monomial map $F_{m+1} : \mathbf{P}^{m+1} \to \mathbf{P}^{M_{m+1}}$ by sending the point $(x_0 : \dots : x_{m+1})$ to the point whose homogeneous coordinates are the values of the distinct monomials in h_{m+1} at that point. The map F_{m+1} is birational because for each j such that $0 \leqslant j \leqslant m+1-d$, the monomials $x_j^{r_{m+1}} x_k$ (for $0 \leqslant k \leqslant m+1$) provide an inverse map to F_{m+1} on the affine open set $x_j \neq 0$. Since $\overline{F_{m+1}(Y_m^d)}$ is irreducible and defined over k, it has an irreducible hyperplane section Z_m^d defined over k by Theorem 2.6. We define p_{m+1} to be the pullback of the hyperplane that cuts out the section Z_m^d. Since the monomials defining the map F_{m+1} are precisely the monomials appearing in the polynomial h_{m+1}, it follows that p_{m+1} has the same monomials as h_{m+1}, but with "generic" coefficients. Now the pullback of Z_m^d is $X_{m+1}^d = V(p_{d+1}, \dots, p_{m+1})$. By Proposition 2.1, the initial complex of X_{m+1}^d has dimension d, so X_{m+1}^d has dimension d also. Since X_{m+1}^d is a complete intersection, all of its irreducible components have dimension d by Theorem 2.5. Now Z_m^d is irreducible, and any additional components introduced by pulling back through F_{m+1} must lie in the subvariety of $\mathbf{P}^{m+1}$ where F_{m+1} is not an isomorphism. But this subvariety is

Table 1
The polynomials g_i in the construction of X_4^1

Polynomial	Weight	Degree
$g_2 = y_0 y_1$	-3	2
$g_3 = y_0 y_1 + y_0 y_2^2 + y_1^2 y_2$	-5	3
$g_4 = y_0 y_1^2 + y_0^2 y_2 + y_1^2 y_2^2 + y_0^2 y_3^2 + y_1^3 y_3 + y_2^4 y_3^2$	-10	6

contained in $V(x_0, \ldots, x_{m+1-d})$, so it has dimension at most $d-1$. Therefore no additional components are created by pulling back through F_{m+1} and so X_{m+1}^d is irreducible. □

This concludes the proof of Theorem 1.1.

Example 2.6.2. We give explicit polynomials defining X_4^1 to illustrate the proof of Theorem 1.1 in the case $d = 1$, $n = 4$.

The polynomials f_i from Proposition 2.1 are as follows:

$$f_2 = x_0 x_1 x_2,$$
$$f_3 = x_0 x_1 x_3 + x_0 x_2 x_3 + x_1 x_2 x_3,$$
$$f_4 = x_0 x_1 x_4 + x_0 x_2 x_4 + x_1 x_2 x_4 + x_0 x_3 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4.$$

We select the weight vector $w = (0, 1, 2, 3, 4)$. We choose the polynomials g_i to have the least negative weight possible, and secondarily to have the least possible degree in the variables y_j. In computing the weights one must remember that $y_j = x_j / x_i$ only in the polynomial g_i. We display the polynomials g_i with their weights and degrees in Table 1. For the given choice of the polynomials g_i, $r_2 = 4$, $r_3 = 6$, and $r_4 = 11$, and so

$$\begin{aligned}
h_2 &= x_0 x_1 x_2^3 + x_1^4 x_2 + x_1^5 + x_1^4 x_0 + x_0^4 x_2 + x_0^4 x_1 + x_0^5, \\
h_3 &= x_0 x_1 x_3^5 + x_0 x_2^2 x_3^4 + x_1^2 x_2 x_3^4 + (x_0 + x_1 + x_2 + x_3)\left(x_0^6 + x_1^6 + x_2^6\right), \\
h_4 &= x_0 x_1^2 x_4^9 + x_0^2 x_2 x_4^9 + x_1^2 x_2^2 x_4^8 + x_0^2 x_3^2 x_4^8 + x_1^3 x_3 x_4^8 + x_2^4 x_3^2 x_4^6 \\
&\quad + (x_0 + x_1 + x_2 + x_3 + x_4)\left(x_0^{11} + x_1^{11} + x_2^{11} + x_3^{11}\right).
\end{aligned}$$

Now $X_1^1 = \mathbf{P}^1$, $Y_1^1 = \mathbf{P}^2$, and F_2 is the rational map from $\mathbf{P}^2$ to $\mathbf{P}^6$ given by

$$(x_0 : x_1 : x_2) \mapsto \left(x_0 x_1 x_2^3 : x_1^4 x_2 : x_1^5 : x_1^4 x_0 : x_0^4 x_2 : x_0^4 x_1 : x_0^5\right). \tag{4}$$

It is undefined at the point $(0 : 0 : 1)$, and $\overline{F_2(Y_1^1)}$ is a toric surface with a $\mathbf{Z}/4\mathbf{Z}$ quotient singularity (type A_3). The exceptional divisor of the inverse map consists of two (-1)-lines intersecting at the singular point. A generic hyperplane section misses the singular point but meets both of the exceptional lines, so the pullback X_2^1 has a node at $(0 : 0 : 1)$. Since it is a plane curve of degree 5, it has genus 5 and is trigonal.

Table 2
The minimal generators of $\mathrm{in}_<(I(X_4^1))$

$x_0x_1x_2^3$	$x_1^3x_2^4x_3^4$	$x_0x_2^5x_3^4$	$x_0x_1x_3^5$	$x_2^{10}x_3^8x_4^6$
$x_1^3x_2^{12}x_4^8$	$x_1^5x_3^{11}x_4^8$	$x_0^3x_3^{14}x_4^8$	$x_0x_1^7x_4^9$	$x_0^2x_2x_4^9$
$x_1^3x_2^{11}x_4^9$	$x_1x_2^8x_3^6x_4^6$	$x_1^4x_2^3x_3x_4^8$	$x_1^3x_2^{10}x_3x_4^8$	$x_1^3x_2^8x_3^2x_4^8$
$x_1^3x_2^6x_3^3x_4^8$	$x_1^2x_2^7x_3^4x_4^8$	$x_0^2x_2^4x_3^6x_4^8$	$x_1^4x_2^2x_3^7x_4^8$	$x_0^3x_2^3x_3^8x_4^8$
$x_1^5x_2x_3^9x_4^8$	$x_0^2x_2^3x_3^9x_4^8$	$x_0^3x_2^2x_3^{10}x_4^8$	$x_0^3x_2x_3^{12}x_4^8$	$x_0x_1^5x_2x_4^9$
$x_0x_1^3x_2^2x_4^9$	$x_1^3x_2^9x_3x_4^9$	$x_1^3x_2^7x_3^2x_4^9$	$x_1^3x_2^5x_3^3x_4^9$	$x_1^5x_2x_3^4x_4^9$
$x_1^4x_2^2x_3^4x_4^9$	$x_1^2x_2^6x_3^4x_4^9$	$x_1^4x_2x_3^5x_4^9$	$x_1^3x_2^3x_3^8x_4^9$	$x_0x_1^2x_2^2x_3^4x_4^9$

Now the cone Y_2^1 has a double line $x_0 = x_1 = 0$ corresponding to the node of X_2^1, with a quintuple point at the vertex of the cone. The map F_3 maps $\mathbf{P}^3$ to $\mathbf{P}^{14}$, and the image $F_3(Y_2^1)$ has a singular line as well. The intersection X_3^1 of Y_2^1 with the pullback of a generic hyperplane section has a tenfold point at $(0:0:0:1)$ and also has a node somewhere on the singular line of Y_2^1. An additional iteration of this process gives the final curve X_4^1.

We refine the weight vector w to a term order $<$ by breaking ties using lexicographic order. Then the initial ideal of the ideal defining X_4^1 in this term order consists of 35 monomials, which are listed in Table 2. Of these, the first 11 have support on only three variables and themselves define the variety A_4^1 set-theoretically. The last of the 11 monomials is redundant because it has the same support as the 6th monomial. Of the remaining 24 monomials, 23 have support on four variables, and the last has support on all five variables.

Given a desired initial complex, if the variety of its Stanley–Reisner ideal can be written as a set-theoretic complete intersection in such a way that the exponents and coefficients can be deformed, arguments like those above should allow the construction of an irreducible variety whose initial complex with respect to an appropriate term order is the desired complex. The author used such techniques to construct term orders and irreducible curves in $\mathbf{P}^3$ with every possible initial complex in [1]. When the desired initial complex cannot easily be written as a set-theoretic complete intersection, a different technique based on splitting and gluing vertices may be used. We discuss this technique in the next section.

3. Unirational varieties

If the variety of the Stanley–Reisner ideal of a desired initial complex cannot be written as a set-theoretic complete intersection, it may be possible to "split" some of the vertices in the desired initial complex to obtain a simpler complex which can be written as an initial complex by known methods. If the chosen term order refines a weight vector in which the split vertices have the same weight, it is possible to "glue" the split vertices back together. It is important that the pieces of a split vertex not be connected in the simpler complex. We will apply the following "gluing" construction to some toric varieties to obtain some initial complexes that do not appear to be set-theoretic complete intersections, in particular the triangulation of $\mathbf{RP}^2$ obtained by identifying antipodal vertices of an icosahedron.

The gluing of points in an initial complex may be described as follows.

Theorem 3.1. *Let $X \subset \mathbf{P}^n$ be a variety and let $\Delta = \Delta_<(X)$ be its initial complex with respect to a term order $<$. Let ω be a weight vector such that $\mathrm{in}_\omega(I(X)) = \mathrm{in}_<(I(X))$. Let $S \subset$ vertices(Δ) such that all $\{v_i, v_j\} \subset S$ satisfy*:

(1) $\omega(v_i) = \omega(v_j)$,
(2) $\{v_i, v_j\}$ *is a nonface of* Δ.

Then there exists a variety $Y \subset \mathbf{P}^m$ and a term order $<'$ such that $\Delta_{<'}(Y)$ is obtained from Δ by identifying the vertices in S.

Proof. We introduce coordinates $x_0, x_1, \ldots, x_n$ on $\mathbf{P}^n$ and $y_0, y_1, \ldots, y_m$ on $\mathbf{P}^m$. Let $\pi : \mathbf{P}^n \to \mathbf{P}^m$ be a zero–one matrix projection that sends the x_i corresponding to vertices outside S to distinct variables y_j (for some $j > 0$) and sends the x_i corresponding to vertices in S to the variable y_0. (Note that the nonzero matrix entries in π can actually be any nonzero element of the ground field; we choose ones here for concreteness.) Let $\pi(\omega)$ be the weight vector that assigns the weight $\omega(x_i)$ to $\pi(x_i)$. This is consistent since the vertices in S all have the same weight. Let $Y = \pi(X)$ and let $<'$ be any term order extending the weight vector $\pi(\omega)$.

Adapting the construction from Lemma 3 of [9] to the projective case, we define the deformation $\widetilde{X}$ of X to its initial ideal with respect to the weight vector ω to be the closure in $\mathbf{P}^n \times \mathbf{A}^1$ of $\{((q^{\omega_0}x_0 : \cdots : q^{\omega_n}x_n), q) \mid (x_0 : \cdots : x_n) \in X, q \in k\}$. There is a natural projection $\widetilde{X} \to \mathbf{A}^1$ with fibers X_q. The fibers X_q are isomorphic to X for $q \neq 0$, and the fiber X_0 is the variety of the initial ideal of $I(X)$ with respect to ω. If we define $\widetilde{Y}$ similarly, then the total space of the deformation $\widetilde{X}$ projects to a subset of the total space of the deformation $\widetilde{Y}$ of Y to its initial ideal with respect to the weight vector $\pi(\omega)$.

We claim that π is a finite morphism. Clearly π is a rational map. Let B_X be the set of points in the total space of $\widetilde{X}$ where π is not a morphism and let B_Y be the set of points in the total space of $\widetilde{Y}$ whose preimage is not finite. Both B_X and B_Y are closed sets. When $q \neq 0$, the fibers X_q (respectively Y_q) of the deformations are all isomorphic to X (respectively Y) and the deformation of X to X_q and Y to Y_q commutes with the projection π from X to Y and from X_q to Y_q. This means that if $x_q \in X_q \cap B_X$ for some $q \neq 0$, then there is a corresponding point $x_{q'} \in X_{q'} \cap B_X$ for every $q' \neq 0$. Since B_X is closed, there must exist a point $x_0 \in X_0 \cap B_X$. Similarly, if $y_q \in Y_q \cap B_Y$ for some $q \neq 0$, then there exists a point $y_0 \in Y_0 \cap B_Y$ since B_Y is closed.

If the projection from the initial variety $X_0 = V(\mathrm{in}_<(I(X)))$ to the initial variety $Y_0 = V(\mathrm{in}_{<'}(I(Y)))$ is not finite or not a morphism, then some projective coordinate subspace contained in X_0 is collapsed to a lower-dimensional projective coordinate subspace in Y_0. Therefore the corresponding simplex in $\Delta_<(I(X)) = \Delta(I(X_0))$ is collapsed to a lower-dimensional simplex in $\Delta_{<'}(I(Y)) = \Delta(I(Y_0))$ by the projection π. In particular, two or more of the vertices of the collapsed simplex belong to S, contradicting the hypothesis.

Since the restriction of π to (the deformation of) X is a finite morphism, we have $\pi(X) = Y$ and so Δ has the expected form. $\square$

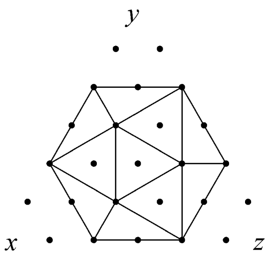

Fig. 1. Triangulation of the embedding polygon of $X \hookrightarrow \mathbf{P}^8$.

Remark 3.1.1. If the restriction of π to X is also generically one-to-one, then the top-dimensional multiplicities are preserved; this is a consequence of the projection formula in intersection theory [3, Proposition 2.3(c)] and the fact that distinct top-dimensional cycles are rationally inequivalent [3, §1.3]. However, the multiplicities of lower-dimensional components may change; in particular, embedded components may appear where none existed before the projection.

Example 3.1.2. We construct an irreducible unirational variety $Y \subset \mathbf{P}^5$ and a term order $<'$ such that $\Delta_{<'}(Y)$ is the icosahedral triangulation of $\mathbf{RP}^2$.

The rational parametrization of the variety Y sends $(x : y : z)$ to:

$$\left(x^2\left(y^4+z^4\right) : y^2\left(x^4+z^4\right) : z^2\left(x^4+y^4\right) : xy^2z^3 : yz^2x^3 : zx^2y^3\right). \tag{5}$$

Let $<'$ be the term order obtained by refining the weight vector $(1, 1, 1, 0, 0, 0)$ using (for definiteness) lexicographic order.

We begin with the toric variety X parametrized by the map that sends $(x : y : z)$ to

$$\left(x^2y^4 : x^2z^4 : y^2x^4 : y^2z^4 : z^2x^4 : z^2y^4 : xy^2z^3 : yz^2x^3 : zx^2y^3\right) \tag{6}$$

and the weight vector is given by $\omega = (1, 1, 1, 1, 1, 1, 0, 0, 0)$. The initial complex $\Delta_\omega(X)$ triangulates the embedding polygon of $X \hookrightarrow \mathbf{P}^8$. We may visualize this triangulation as in Fig. 1 by assigning the barycentric coordinates $(2, 4, 0), (2, 0, 4), \ldots, (2, 3, 1)$ to the vertices of the triangulation, corresponding to the monomials defining the map (6). We apply Theorem 3.1 three times, once to each pair of antipodal vertices of the outer hexagon. Making these identifications gives the desired initial complex.

As noted in the proof of Theorem 3.1, the initial complex of the variety Y in Example 3.1.2 is independent of the choice of nonzero entries in the matrix of the projection $\pi : X \to Y$. If the identification of the vertices in the subset S collapses a simplex to a simplex of lower dimension (so that the hypothesis of Theorem 3.1 does not apply), the resulting initial complex may differ from that predicted by the proposition, and may also depend on the choice of the nonzero entries in the projection matrix. We illustrate this in the next example.

Example 3.1.3 (*Gräbe* [5]). We construct varieties X and Y and term orders $<$ and $<'$ so that the initial complex $\Delta_{<'}(I(Y))$ does not satisfy the conclusion of Theorem 3.1 for a specific choice of the coefficients.

In Gräbe's example Y is parametrized by the map

$$(s:t:u) \mapsto \bigl(s^3 : s^2t : stu : su(u-s) : u^2(u-s)\bigr) \tag{7}$$

and $<'$ is the lexicographic ordering. In fact we may also take $<'$ to be any refinement of the weight vector $(6, 4, 2, 1, 0)$.

We will reconstruct the variety Y by gluing two separate pairs of vertices in one step. To adhere as closely as possible to the hypotheses of Theorem 3.1, we should really glue the first pair and the second pair in separate steps; but this would not change the final outcome. Performing a construction similar to that of Theorem 3.1 with the variety X parametrized by the map

$$(s:t:u) \mapsto \bigl(s^3 : s^2t : stu : su^2 : s^2u : u^3 : su^2\bigr) \tag{8}$$

and the weight vector $\omega = (6, 4, 2, 1, 1, 0, 0)$ yields the variety Y and the term order $<'$ above, provided we choose the nonzero entries in the projection π correctly (see below). The variety X would be toric except that the fourth and seventh monomial parameters are the same. Its initial complex consists of the simplices $\{x_0, x_1, x_4\}$, $\{x_1, x_2, x_4\}$, $\{x_2, x_4, x_6\}$, and $\{x_2, x_5, x_6\}$. If we construct a variety Y' by the same method but using generic nonzero entries in π, the last simplex collapses, and $\Delta_{<'}(Y')$ satisfies the conclusion of Theorem 3.1.

However, in Gräbe's example the choice of nonzero matrix entries $(1, 1, 1, 1, -1, 1, -1)$ causes an extra simplex to appear. These nonzero matrix entries give the projection matrix of π as:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

In addition to the expected simplices $\{y_0, y_1, y_3\}$, $\{y_1, y_2, y_3\}$, and $\{y_2, y_3, y_4\}$, we get a fourth simplex $\{y_0, y_2, y_4\}$. Where does this fourth simplex come from? One explanation is that the specific choice of nonzero matrix entries causes a second branch of the surface to fall on the "heaviest" point $(1, 0, 0, 0, 0)$. The second branch then contributes its own simplex to the initial complex instead of being pushed out to $\{y_2, y_3, y_4\}$.

4. Questions

Can the behavior of the gluing construction be predicted when some simplices are collapsed to lower dimensions? In some cases a flip occurs in the unglued simplicial complex that prevents the simplices from collapsing.

What initial complexes are possible for general irreducible varieties? Although [13] and the present paper make progress toward answering this question, it is still open in general. The limitations of the techniques presented here suggest the following example.

Let $\varGamma_{24}^2$ be the 2-skeleton of the regular (4-dimensional) 24-cell. To reduce the number of vertices, we identify antipodal vertices of the 24-cell to obtain $\varGamma_{12}^2$. Let B_{12}^2 be the corresponding arrangement of 2-planes. It has codimension 9 in $\mathbf{P}^{11}$ and can be defined set-theoretically by just 10 polynomials. Unfortunately, $10 > 9$ and it seems unlikely that B_{12}^2 can be written as a set-theoretic complete intersection.

Every edge of $\varGamma_{12}^2$ has 3 triangles touching it, so each of its vertices would have to be glued if the "gluing" starts with a toric variety. However, it is generally very difficult to glue boundary vertices to interior vertices because of the requirement that they be assigned the same weight. Therefore it seems likely that every vertex would have to be a boundary vertex. Even then, the particular triangulation would impose many restrictions on the possible weights.

The techniques of [13] also fail on $\varGamma_{12}^2$ since B_{12}^2 has codimension 9. It seems likely that a new technique would be needed to deal with this example.

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